Oxford Cambridge and RSA

Accredited

A Level Further Mathematics A

Y545 Additional Pure Mathematics

Sample Question Paper

Version 2

Date – Morning/Afternoon

Time allowed: 1 hour 30 minutes

You must have:
Printed Answer Booklet
Formulae A Level Further Mathematics A

You may use:
- a scientific or graphical calculator

INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by g m s^{-2}. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

- The total number of marks for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Answer **all** the questions.

1 A curve is given by $x = t^2 - 2\ln t$, $y = 4t$ for $t > 0$. When the arc of the curve between the points where $t = 1$ and $t = 4$ is rotated through 2π radians about the *x*-axis, a surface of revolution is formed with surface area *A*.
Given that $A = k\pi$, where *k* is an integer,

- write down an integral which gives *A* and
- find the value of *k*. **[4]**

2 Find the volume of tetrahedron OABC, where O is the origin, A = (2, 3, 1), B = (–4, 2, 5) and C = (1, 4, 4). **[3]**

3 Given $z = x\sin y + y\cos x$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} + z = 0$. **[5]**

4 **(i)** Solve the recurrence relation $u_{n+2} = 4u_{n+1} - 4u_n$ for $n \geq 0$, given that $u_0 = 1$ and $u_1 = 1$. **[4]**

(ii) Show that each term of the sequence $\{u_n\}$ is an integer. **[2]**

5 **In this question you must show detailed reasoning.**
It is given that $I_n = \int_0^{\pi} \sin^n \theta \,\mathrm{d}\theta$ for $n \geq 0$.

(i) Prove that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$. **[5]**

(ii) **(a)** Evaluate I_1. **[2]**

(b) Use the reduction formula to determine the exact value of $\int_0^{\pi} \cos^2\theta \, \sin^5\theta \,\mathrm{d}\theta$. **[2]**

6 A surface *S* has equation $z = \mathrm{f}(x, y)$, where $\mathrm{f}(x, y) = 2x^2 - y^2 + 3xy + 17y$. It is given that *S* has a single stationary point, *P*.

(i) **(a)** Determine the coordinates of *P*. **[5]**

(b) Determine the nature of *P*. **[3]**

(ii) Find the equation of the tangent plane to *S* at the point $Q(1, 2, 38)$. **[2]**

7 In order to rescue them from extinction, a particular species of ground-nesting birds is introduced into a nature reserve. The number of breeding pairs of these birds in the nature reserve, t years after their introduction, is an integer denoted by N_t . The initial number of breeding pairs is given by N_0 .

An initial discrete population model is proposed for N_t .

$$\text{Model I: } N_{t+1} = \tfrac{6}{5} N_t \left(1 - \tfrac{1}{900} N_t\right)$$

(i) (a) For Model I, show that the steady state values of the number of breeding pairs are 0 and 150. **[3]**

(b) Show that $N_{t+1} - N_t < 150 - N_t$ when N_t lies between 0 and 150. **[3]**

(c) Hence find the long-term behaviour of the number of breeding pairs of this species of birds in the nature reserve predicted by Model I when $N_0 \in (0, 150)$. **[2]**

An alternative discrete population model is proposed for N_t .

$$\text{Model II: } N_{t+1} = \text{INT}\left(\tfrac{6}{5} N_t \left(1 - \tfrac{1}{900} N_t\right)\right)$$

(ii) (a) Given that $N_0 = 8$, find the value of N_4 for each of the two models. **[2]**

(b) Which of the two models gives values for N_t with the more appropriate level of precision? **[1]**

8 The set X consists of all 2×2 matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where x and y are real numbers which are not **both** zero.

(i) (a) The matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ are both elements of X.

Show that $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}=\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ for some real numbers p and q to be found in terms of a, b, c and d. **[2]**

(b) Prove by contradiction that p and q are not **both** zero. **[5]**

(ii) Prove that X, under matrix multiplication, forms a group G.
[You may use the result that matrix multiplication is associative.] **[4]**

(iii) Determine a subgroup of G of order 17. **[2]**

9 **(i) (a)** Prove that $p\equiv\pm1\pmod 6$ for all primes $p>3$. **[2]**

(b) Hence or otherwise prove that $p^2-1\equiv 0\pmod{24}$ for all primes $p>3$. **[3]**

(ii) Given that p is an odd prime, determine the residue of 2^{p^2-1} modulo p. **[4]**

(iii) Let p and q be distinct primes greater than 3. Prove that $p^{q-1}+q^{p-1}\equiv 1\pmod{pq}$. **[5]**

END OF QUESTION PAPER